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Ch. III Bounded operators.

III. 1.1. Operator topologies

$\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$; $\mathcal{L}(\mathcal{H})$ if

$\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{H}$ - a Hilbert space.

- Uniform (or norm) operator topology
- defined by the norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_1}.$$

- Strong operator topology

- defined by the base of neighborhoods of 0:

$$\left\{ T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \mid \|Tx_i\| < \varepsilon, i = 1, \dots, n \right\} = \bigcup_{x_1, \dots, x_n} (\varepsilon), \quad x_1, \dots, x_n \in \mathcal{B}_1,$$

$$s\text{-}\lim_{n \rightarrow \infty} T_n = T \Leftrightarrow \lim_{n \rightarrow \infty} T_n x = Tx$$

$$\forall x \in \mathcal{B}_1.$$

• Weak operator topology

- defined by the base of neighborhoods of 0:

$$\{ T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \mid |l_i(Tx_j)| < \varepsilon, \\ i = 1, \dots, n; j = 1, \dots, m \}$$

$$= \bigcup_{x_1, \dots, x_n; l_1, \dots, l_m} x_j \in \mathcal{B}_1, l_i \in \mathcal{B}_2^*$$

$$w\text{-}\lim_{n \rightarrow \infty} T_n = T \Leftrightarrow \lim_{n \rightarrow \infty} l(T_n x)$$

$$= l(Tx), \quad \forall x \in \mathcal{B}_1, l \in \mathcal{B}_2^*$$

"U > S > W"

Thm Let $\{T_n\} \subset \mathcal{L}(\mathcal{H})$.

(i) If $\exists \lim_{n \rightarrow \infty} (T_n x, y) \quad \forall x, y \in \mathcal{H}$,

then $\exists T \in \mathcal{L}(\mathcal{H})$ s.t. $w\text{-}\lim_{n \rightarrow \infty} T_n$

$$= T$$

(ii) If $\lim_{n \rightarrow \infty} T_n x \exists \quad \forall x \in \mathcal{H}$,

then $\exists T \in \mathcal{L}(\mathcal{H})$ s.t.

$$s\text{-}\lim_{n \rightarrow \infty} T_n = T.$$

III. 2 Adjoints

$T: \mathcal{B}_1 \rightarrow \mathcal{B}_2$, bounded \Rightarrow

$\exists T': \mathcal{B}_2^* \rightarrow \mathcal{B}_1^*$, bounded.

Namely, $(T'\ell)(x) = \ell(Tx)$
 $\forall x \in \mathcal{B}_1$ and $\ell \in \mathcal{B}_2^*$.

Ex. $\mathcal{B}_1 = \mathcal{B}_2 = \ell_1$, $T(x_1, x_2, \dots) = (0, x_1, \dots)$

$T'(y_1, y_2, \dots) = (y_2, y_3, \dots)$.

Thm The map $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \rightarrow \mathcal{L}(\mathcal{B}_2^*, \mathcal{B}_1^*)$,
 is an isometry. given by $T \mapsto T'$

Proof

$$\begin{aligned}
 \|T\| &= \sup_{\|x\|=1} \|Tx\| \\
 &= \sup_{\|x\|=1} (\sup_{\|\ell\|=1} |\ell(Tx)|) \quad (*) \\
 &= \sup_{\|\ell\|=1} (\sup_{\|x\|=1} |(T'\ell)(x)|) \\
 &= \sup_{\|\ell\|=1} \|T'\ell\| \\
 &= \|T'\|.
 \end{aligned}$$

(*) uses Corollary of Hahn-Banach thm.

The Hilbert space adjoint

- Hermitian conjugate.

C: $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^*$ - complex anti-linear;

$$T^* = \bar{C} T' C$$

$$\Leftrightarrow (Tx, y) = (x, T^*y)$$

$\forall x, y \in \mathcal{H}$.

Def $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint if $T^* = T$.

Thm (a) $T \rightarrow T^*$ is a complex anti-linear isomorphism $\mathcal{L}(\mathcal{H}) \xrightarrow{\sim} \mathcal{L}(\mathcal{H})$

$$(b) (ST)^* = T^* S^*$$

$$(c) (T^*)^* = T$$

(d) If $\exists T^{-1} \in \mathcal{L}(\mathcal{H})$, then

$$(T^*)^{-1} = (T^{-1})^*$$

(e) $T \rightarrow T^*$ is U & W continuous,
but not S-continuous

$$(d) \|T^*T\| = \|T\|^2.$$

(If $T = T^*$, then

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)|)$$

Def $P \in \mathcal{L}(\mathcal{H})$ is called a projector, if $P^2 = P$. If $P = P^*$, P is called an orthogonal projector.

Fact If $P^2 = P$, then $\text{Im } P$ is closed: if, in addition, $P = P^*$, then

$$P|_{\text{Im } P^\perp} = 0.$$

($Px_n \rightarrow y \Leftrightarrow P^2x_n \rightarrow Py$, i.e. $y = Px$;

$(Px, y) = (\alpha, Py) = 0 \quad \forall y \in \mathcal{H}$
 $\& \alpha \in \text{Im } P^\perp$).

III. 1.2 The spectrum

Def $\lambda \in \mathbb{C}$ is a regular point for $T \in \mathcal{L}(V)$, if $\lambda I - T$ is a bijection (and hence has a bounded inverse).

The bounded operator

$$R_\lambda(T) = (\lambda I - T)^{-1}$$

is called the resolvent of T at λ .

The set of regular points $\rho(T) \subset \mathbb{C}$ (the regular or resolvent set).

Then the spectrum $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Remark $U \in \mathcal{L}(\mathcal{H})$ is an isometry

if $\|Ux\| = \|x\| \quad \forall x \in \mathcal{H}$.

U is isometry $\Leftrightarrow (Ux, Uy) = (x, y)$

$\forall x, y \in \mathcal{H}$; this implies

$$\text{Ker } U = \{0\}.$$

But $\text{Im } U$ is not necessarily \mathcal{H} .

Def U is unitary if U is isometry
and $\text{Im } U = \mathcal{H}$.

Lemma (i) U is isometry $\Leftrightarrow U^*U = I$

(ii) U is unitary $\Leftrightarrow U^*U = UU^* = I$.

Proof

(i) - clear; (ii) U is unitary \Rightarrow

U^{-1} is, i.e., $(U^{-1})^*U^{-1} = I$, so

$UU^* = I$. Conversely, if $UU^* = I$, then U is onto.

Remark 1 If U is onto and

$(Ux, Uy) = (x, y) \quad \forall x, y \in \mathcal{H}$,
then U is linear.

$$(Ux, y) = (x, U^{-1}y).$$

If $T \in L(\mathcal{H})$, then $\text{Ker } T$ is closed and $\text{Im } T$ is not necessarily closed.

Lemma If U is an isometry, then $\text{Im } U$ is closed.

Proof $Ux_n \rightarrow y$, so that

$\{Ux_n\}$ is a Cauchy sequence,

$\|Ux\| = \|x\|$, so $\{x_n\}$ is, i.e.

$x_n \rightarrow x$ and $y = Ux$.

Set P be ^{orthogonal} projector on $\text{Im } U$.

Then $U : \mathcal{H} \rightarrow \text{Im } U$

$$U^*U = I, \quad UU^* = P$$

Ex. $U(a_1, \dots) = (0, a_1, a_2, \dots)$

- isometry

$$U^*(a_1, \dots) = (a_2, \dots);$$

$$U^*U = I, \quad UU^* = P,$$

where

$$P(a_1, a_2, \dots) = (0, a_2, \dots).$$

Remark Here V is just a normed vector space.

Def Let $T \in \mathcal{L}(V)$.

(a) If $\lambda \in \mathbb{C}$ is s.t.

$$Tx = \lambda x, \quad x \in V,$$

then λ is an e-value \in point spectrum = set of all e-values

(b) If λ is not an e-value and if

$\text{Im}(\lambda I - T)$ is not dense, then $\lambda \in$

residual spectrum. Lecture 19 10/17/03

Analytic functions in Banach spaces

$$\mathbb{C} \supset D \ni z \mapsto x(z) \in \mathcal{B}$$

Def $x: D \rightarrow \mathcal{B}$ is

(a) strongly analytic in D if

$$\exists \lim_{h \rightarrow 0} \frac{x(z+h) - x(z)}{h} \quad \forall z \in D$$

(b) is weakly analytic if

$$l(x): D \rightarrow \mathbb{C}$$

is analytic for every $l \in \mathcal{B}^*$.

Then Every weakly analytic function in a Banach space is strongly analytic.

Proof Let $x: D \rightarrow \mathcal{B}$ be weakly analytic. Then for every $z_0 \in D$ and every $\ell \in \mathcal{B}^*$

$$\ell\left(\frac{x(z_0+h) - x(z_0)}{h}\right) - \frac{d}{dz}(\ell(x)(z_0))$$

$$= \frac{1}{2\pi i} \oint_{\mathcal{C}} \left[\frac{1}{h} \left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \right]$$

$$\ell(x)(z) dz,$$

where $\mathcal{C} = \{ |z-z_0| = r \} \subset D$ and contains $z_0 + h$ inside.

Now $|\ell(x)(z)| \leq C_\ell$ on \mathcal{C} .

Consider for every $z \in \mathcal{C}$ $x(z): \mathcal{B}^* \rightarrow \mathbb{C}$;
this family is bounded on every $\ell \in \mathcal{B}^*$
 \Rightarrow by the UBT,

$$\sup_{z \in \mathcal{C}} \|x(z)\| \leq C < \infty.$$

$$z \in \mathcal{C}$$

Thus

$$\left\{ \ell\left(\frac{x(z_0+h) - x(z_0)}{h}\right) - \frac{d}{dz} \ell(x)(z_0) \right\} \\ \leq \frac{\|\ell\|}{2\pi} \sup_{z \in \mathcal{C}} \|x(z)\| \oint_{\mathcal{C}} \left| \frac{1}{(z-z_0-h)(z-z_0)} - \frac{1}{(z-z_0)^2} \right| dz$$

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and the sequence $\frac{x(z_0+h) - x(z_0)}{h} = x_h$
has the property that

x_h is Cauchy uniformly for
all $l \in \mathcal{B}^*$, $\|l\| \leq 1$. Then by
the lemma below x_h is Cauchy in
 \mathcal{B} , so that $x(z)$ is strongly analytic.

Lemma The sequence $\{x_n\} \subset \mathcal{B}$
is Cauchy iff $\{l(x_n)\}$ is Cauchy,
uniformly for $\|l\| \leq 1$, $l \in \mathcal{B}^*$.

Proof Indeed,

$$\|x_n - x_m\| = \sup_{\|l\| \leq 1} |l(x_n - x_m)|.$$

Thm Let $T \in \mathcal{L}(\mathcal{B})$. Then
 $\rho(T)$ is open and $R_\lambda(T)$ is
analytic on $\rho(T)$. For every λ, μ
 $\in \rho(T)$

$$R_\lambda(T) - R_\mu(T) = (\bar{\lambda} - \mu) R_\lambda(T) \cdot R_\mu(T) \quad (\text{the Hilbert identity})$$

so that $R_\lambda(T)$ and $R_\mu(T)$
commute.

Corollary $\sigma(T) \neq \emptyset$.

Proof For $|\lambda| > \|T\|$ we have the Neumann series

$$R_\lambda(T) = \frac{1}{\lambda - T} = \frac{1}{\lambda} \left(1 - \frac{T}{\lambda}\right)^{-1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{\lambda} \left(\frac{T}{\lambda}\right)^n,$$

convergent in \mathcal{U} topology. If $\sigma(T) = \emptyset$, then $R_\lambda(T)$ would be entire function s.t.

$$\lim_{|\lambda| \rightarrow \infty} R_\lambda(T) = 0,$$

so by Liouville's theorem, $R_\lambda(T) = 0$.

Proof of the theorem.

$$\frac{1}{\lambda - T} = \frac{1}{\lambda - \lambda_0 + \lambda_0 - T} = \frac{1}{\lambda_0 - T} \cdot \frac{1}{\left(1 + \frac{\lambda_0 - \lambda}{\lambda_0 - T}\right)}$$

Claim

$$R_\lambda(T) = R_{\lambda_0}(T) \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}(T)^n$$

if $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}(T)\|}$.

Indeed, for these λ the series is convergent
a multiplying the series by
 $(\lambda I - T)$ gives I .

(Note that $\|B^n\| \leq \|B\|^n$).

Thus $\left\{ |\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}(T)\|} \right\} \subset g(T)$,

$g(T)$ is open and $R_\lambda(T)$ is analytic
on $g(T)$.

(Note that

$$\begin{aligned} (\lambda - T) R_{\lambda_0}(T) &= (\lambda - \lambda_0) R_{\lambda_0}(T) + I \\ &= I - (\lambda_0 - \lambda) R_{\lambda_0}(T) \end{aligned}$$

Finally,

$$\begin{aligned} R_\lambda(T) - R_\mu(T) &= R_\lambda(T)(\mu - T) R_\mu(T) \\ &\quad - R_\lambda(T)(\lambda - T) R_\mu(T) \\ &= (\mu - \lambda) R_\lambda(T) R_\mu(T). \end{aligned}$$

Def $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$

$$\lambda \in \sigma(T)$$

- spectral radius of T , $r(T) \leq \|T\|$.

Thm 1 Let $T \in \mathcal{L}(\mathcal{B})$. Then $\frac{10/20/03}{}$

$$r(T) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T^n\|}.$$

If $A = A^*$, $r(A) = \|A\|$.

Proof (different from R-S)
The series ^{slightly}

$$\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$
 has radius of convergence $R = \overline{\lim} \|T^n\|^{\frac{1}{n}},$
so that for

$|\lambda| > R$ its sum $= R_\lambda(T) \Rightarrow$

$r(T) = R$. (If $r(T) = R' < R$,

then $R_\lambda(T)$ will be analytic in $|\lambda| > R'$ - contradiction.)

But $\lambda \in \sigma(T) \Rightarrow \lambda^n \in \sigma(T^n)$.

Indeed, if $\lambda^n - T^n$ is invertible, then

$$\lambda^n - T^n = P_{n-1}(T)(\lambda - T)$$

implies that $\lambda - T$ is also invertible.

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Thus $r(T) \leq \|T^n\|^{1/n}$, i.e.

$$r(T) \leq \lim \sqrt[n]{\|T^n\|}.$$

If $A = A^*$, then $\|A^2\| = \|A\|^2, \dots, \|A^{2^n}\| = \|A\|^{2^n}$ and $r(A) = \|A\|$.

Thm 2. If $T \in \mathcal{L}(B)$, then

$$\sigma(T) = \sigma(T') \text{ and } R_\lambda(T') = R_\lambda(T)'.$$

If $T \in \mathcal{L}(H)$, then

$$\sigma(T^*) = \overline{\sigma(T)} \text{ and } \left. \begin{array}{l} R_\lambda(T^*) = R_{\bar{\lambda}}(T)^* \\ \end{array} \right\} \text{ prove this}$$

Ex. $T \in \mathcal{L}(l_1)$,

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots)$$

$$T' \in \mathcal{L}(l_\infty),$$

$$T'(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$$

$$\|T\| = \|T'\| = 1$$

($\text{Im}(\lambda I - T)$)
not dense

	Spectrum	Point Sp.	Residual Sp
T	$ \lambda \leq 1$	$ \lambda < 1$	\emptyset
T'	$ \lambda \leq 1$	\emptyset	$ \lambda < 1$

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Proof of the proposition for $\mathcal{B} = \mathcal{H}$.
(actually, Thm 2)

Claim $\rho(T) \subset \overline{\rho(T^*)}$.

Indeed, let $\lambda \in \rho(T)$. Then

$T^* - \bar{\lambda}$ is 1-1. Suppose $\text{Ker}(T^* - \bar{\lambda}) \neq \{0\}$,
 $(T^* - \bar{\lambda})x = 0$, $x \neq 0$. Then $\forall y \in \mathcal{H}$

$$0 = (y, (T^* - \bar{\lambda})x) = ((T - \lambda)y, x),$$

i.e. $x \perp \text{Im}(T - \lambda) = \mathcal{H}$ - a contradiction.

Next, $T^* - \bar{\lambda}$ is onto. First,

$\text{Im}(T^* - \bar{\lambda})$ is closed. Let $K^*_{\underbrace{x_n}_{K^*}} \rightarrow y$
in \mathcal{H} . Then

$$\|x_n - x_m\|^2 = (K^*(x_n - x_m), K^{-1}(x_n - x_m))$$

$$\leq \|K^*(x_n - x_m)\| \|K^{-1}(x_n - x_m)\|,$$

i.e.

$$\|x_n - x_m\| \leq \|K^{-1}\| \|K^*_{x_n} - K^*_{x_m}\|$$

$\Rightarrow \{x_n\}$ - Cauchy, $x_n \rightarrow x$ &

$$y = K^*_{\underline{x}}$$

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Second, let $y \perp \text{Im}(T^* - \lambda)$, so
that $\forall x \in \mathcal{H}$

$$0 = (y, (T^* - \bar{\lambda})x) = ((T - \lambda)y, x),$$

so that

$$Ty = \lambda y.$$

Since $\lambda \in g(T)$, $y = 0$.

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To summarize, we proved that

$$\text{Ker } A^* \perp \text{Im } A$$

or, more precisely

$$(\text{Im } A)^\perp = \text{Ker } A^*$$

for $A \in \mathcal{L}(\mathcal{H})$.

(a) $Tx_\lambda = \lambda x_\lambda$, $x_\lambda = (1, \lambda, \lambda^2, \dots)$;

$|\lambda| < 1 \Rightarrow x_\lambda \in l_1$,

$\sigma(T) = \sigma(T') = \{ |\lambda| \leq 1 \}$.

Also, if $T' \xi = \lambda \xi$, then $\xi = 0$. This establishes the first two columns in the table.

Next, λ with $|\lambda| = 1$ is not an eigenvalue for T . Suppose that

$$\overline{\text{Im}(\lambda I - T)} \neq l_1$$

$$\Rightarrow \exists L \in l_\infty \text{ s.t. } \forall x \in l_1$$

$$0 = L((\lambda - T)x) = ((\lambda - T')(L))(x)$$

$\Rightarrow \lambda L = T'L$, λ is an eigenvalue for T' - a contradiction.

Finally, for $|\lambda| < 1$ and $\forall L \in l_\infty$

$$0 = L((T - \lambda)(x_\lambda))$$

$$= (T' - \lambda)(L)(x_\lambda),$$

so that $\overline{\text{Im}(T' - \lambda)} \neq l_\infty$.

Proof that $|\lambda| = 1 \notin$ residual spectrum of $T' - (R - S')$

Prop. Let $T \in \mathcal{L}(\mathcal{B})$.

(a) If $\lambda \in$ residual spectrum of T
 $\Rightarrow \lambda \in$ point spectrum of T' .

(b) If $\lambda \in$ point spectrum of $T \Rightarrow$
 $\lambda \in$ point or residual spectrum of T' .

Proof

In both cases, follows from the example before. Subtle point:

λ is an eigen-value $\nrightarrow \overline{\text{Im}(T-\lambda I)}$
 $\neq \mathcal{B}$ (does not necessarily imply)

Ex. $T(a_1, a_2, \dots) = (a_2, a_3, \dots)$

$(1, 0 \dots 0)$ is an eigen-vector with
 $\lambda = 0$, but $\overline{\text{Im } T} = \mathcal{B}$.

Thm 3 Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then

(a) $\sigma(T) \subset \mathbb{R}$

(b) A has no residual spectrum

(c) eigenvectors corresponding to different eigenvalues are orthogonal.

Proof (arguments in R-S are not quite correct).

Clearly, (a) \Leftrightarrow (b) (by proposition, since discrete and residual spectra are disjoint).

Proof of part (a)

- If λ is an eigen-value, then $\lambda \in \mathbb{R}$:

$$Ax = \lambda x,$$

$$\lambda \|x\|^2 = (Ax, x) = (x, Ax) = \bar{\lambda} \|x\|^2.$$

- If $\overline{\text{Im}(A - \lambda I)} \neq \mathcal{H}$, then λ is an eigenvalue. Indeed, otherwise $\exists y \in \mathcal{H}$ s.t. $\forall x \in \mathcal{H}$

$$(y, (A - \lambda)x) = ((A - \bar{\lambda})y, x) = 0,$$

$$Ay = \bar{\lambda}y \quad \& \quad \lambda = \bar{\lambda}.$$

- λ is an eigenvalue iff $A - \lambda I$ is 1-1.

Now $\lambda \notin \mathbb{C} \setminus \mathbb{R}$ is not an eigenvalue
 $\Rightarrow A - \lambda I$ is 1-1 and on the back
 $\text{Im}(A - \lambda I) = \mathcal{H}$.

Claim $\text{Im}(A - \lambda I)$ is closed.

$$\lambda = \xi + iy,$$

$$\|(A - \lambda I)x\|^2 = \|(A - \xi I)x\|^2$$

$$+ |y|^2 \|x\|^2,$$

$$\|(A - \lambda I)x\| \geq |y| \|x\|.$$

If $y_n = (A - \lambda_n I)x_n$ is Cauchy,
then x_n is Cauchy, $x_n \rightarrow x$

& $y_n \rightarrow y = (A - \lambda I)x$.

$$a^* b = ab$$

$$(a^* b)^* = b^* a^* \text{ and } (a^* b)^* = a^* b$$

$$a^* a = a a^* = I$$

$$a^* a = a a^* = I$$

$$I = (a^*, (A - \lambda I)) = (a^* A, \lambda)$$

$$a^* a = a a^* = I$$

Ex. $\mathcal{H} = L^2(0,1)$,

$$(Qf)(x) = xf(x);$$

$Q \in \mathcal{L}(\mathcal{H})$, $Q = Q^*$ and $\|Q\| \leq 1$.

Claim $\sigma(Q) = [0, 1]$

Indeed, if $\lambda \notin [0, 1]$

$$\left(\frac{1}{\lambda - Q} f \right)(x) = \frac{f(x)}{x - \lambda} \in L^2(0,1)$$

$$\text{since } \left\| \frac{1}{x - \lambda} \right\|^2 = \frac{1}{(x - \operatorname{Re}\lambda)^2 + \operatorname{Im}\lambda^2}$$

$$\leq \frac{1}{(\operatorname{Im}\lambda)^2} \text{ if } \lambda \notin \mathbb{R}$$

$$\leq \frac{1}{\min\{|\operatorname{Re}\lambda|^2, (1 - |\operatorname{Re}\lambda|)^2\}} \text{ if } \lambda \in \mathbb{R} \setminus [0, 1].$$

So $0 \in \sigma(Q)$; $\overline{\operatorname{Im}Q} = \mathcal{H}$

(otherwise 0 would be an eigenvalue),

but $\operatorname{Im}Q \neq \mathcal{H}$, since

$$1 \notin \operatorname{Im}Q \quad (\frac{1}{x} \notin L^2(0,1)).$$

Repeats proof of Thm 2.

III.1.3 Positive operators and polar decomposition

Def $B \in \mathcal{L}(\mathcal{H})$ is positive if
 $(Bx, x) \geq 0 \quad \forall x \in \mathcal{H}$

Fact If \mathcal{H} is \mathbb{C} and $B \geq 0$ then

$$B = B^*$$

Proof $B = A$

$$(Ax, y) + (Ay, x)$$

$$= \frac{1}{2} \left[(A(x+y), x+y) - (A(x-y), x-y) \right]$$

$$\textcircled{!} = \frac{1}{2} \left[(x+y, A(x+y)) - (x-y, A(x-y)) \right]$$

$$= (x, Ay) + (y, Ax),$$

so that

$$(Ax, y) - \overline{(Ax, y)} = (x, Ay) - \overline{(x, Ay)}$$

but $y \mapsto iy$

$$(Ax, y) + \overline{(Ax, y)} = (x, Ay) + \overline{(x, Ay)}$$

so that $(Ax, y) = (x, Ay) \quad \forall x, y \in \mathcal{H}$.

$A \in \mathcal{L}(\mathcal{H})$, A^*A is positive.

Lemma $\sqrt{1-z}$ has power series absolutely convergent for $|z| \leq 1$.

Proof

$$\sqrt{1-z} = \sum_{n=0}^{\infty} c_n z^n, \quad c_0 = 1,$$

all $c_n < 0$ for $n \geq 1$.

$$\begin{aligned} \sum_{n=0}^N |c_n| &= 2 - \sum_{n=0}^N c_n \\ &= 2 - \lim_{x \rightarrow 1^-} \sum_{n=0}^N c_n x^n \end{aligned}$$

$$\leq 2 - \lim_{x \rightarrow 1^-} \sqrt{1-x} = 2, \text{ so that}$$

$$\sum_{n=0}^{\infty} c_n z^n \text{ converges absolutely on } |z|=1.$$

Thm! (The square root lemma) Let

$A \in \mathcal{L}(\mathcal{H})$, $A \geq 0$. Then $\exists! B \in \mathcal{L}(\mathcal{H})$ s.t. $B^2 = A$ & $B \geq 0$. Moreover $BC = CB$ for every $C \in \mathcal{L}(\mathcal{H})$ s.t. $CA = AC$.

Better proof of the Lemma:

$$\sqrt{1-z} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n z^n$$

$$|c_n| = \frac{(n-3/2) \cdots \frac{1}{2} \cdot \frac{1}{2}}{n!}$$

$$\lambda_n = \frac{|c_n|}{|c_{n-1}|} = 1 - \frac{3}{2} \frac{1}{n}.$$

By inequality $(1+x)^{-\alpha} > 1 - \alpha x$

$$\begin{aligned} [f(x) &= (1+x)^{-\alpha} + \alpha x - 1, f'(x) \\ &= -\alpha(1+x)^{-\alpha-1} + \alpha > 0 \text{ for } x > 0] \end{aligned}$$

we have

$$\lambda_n < \frac{1}{\left(1 + \frac{1}{n}\right)^{3/2}} = \frac{n^{3/2}}{(n+1)^{3/2}},$$

i.e.

$$|c_n| \leq \frac{1}{(n+1)^{3/2}} - \text{convergence on}$$

$$|z|=1$$

Proof Consider $\|A\| \leq 1$

$$\|I-A\| = \sup |((I-A)x, x)| \leq 1$$
$$|x|=1$$

(using $A \geq 0$, more specifically,
 $0 \leq I-A \leq I$)

$$\text{Set } B = \sum_{n=0}^{\infty} c_n (I-A)^n$$

-converges in norm $\Rightarrow B^2 = I - (I-A)$
= A by rearranging the terms. Commute
with any C s.t. $AC = CA$.
Moreover, for $\|x\| = 1$

$$(Bx, x) = 1 + \sum_{n=1}^{\infty} c_n ((I-A)^n x, x)$$
$$\geq 1 + \sum_{n=1}^{\infty} c_n = 0 \quad (\text{since } c_n < 0, n \geq 1)$$

and $0 \leq ((I-A)^n x, x) \leq 1$.

Uniqueness $B'^2 = A$, $B' \geq 0$

$B'A = B'^3 = AB' \Rightarrow B'B = BB'$ as
well.

$$B'^2 = B^2 \Rightarrow (B' - B)(B + B')(B' - B) = 0$$
$$\Rightarrow (B' - B)B(B' - B) = (B' - B)B'(B' - B)$$
$$= 0 \quad (\text{sum of } \geq 0 = 0)$$

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Proof that if $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$, then

$$\|A\| = \sup \{|(Ax, x)|\}.$$

$$\|x\|=1$$

It is clear that $\sup \overset{\alpha}{\underset{\|x\|=1}{}} \leq \|A\|$. Conversely,

$$|(Ax, x)| \leq \alpha \|x\|^2 \quad \forall x \in \mathcal{H},$$

so that

$$|A(x+y, x+y)| = |(Ax, x)$$

$$\pm 2 \operatorname{Re}(Ax, y) + (Ay, y)| \leq \alpha \|x+y\|^2.$$

$$(\text{using } (Ay, x) = (y, Ax) = \overline{(Ax, y)})$$

Thus

$$|4 \operatorname{Re}(Ax, y)| \leq \alpha (\|x+y\|^2 + \|x-y\|^2)$$
$$= 2\alpha (\|x\|^2 + \|y\|^2),$$

or, for all $t \in \mathbb{R}$

$$2\alpha t^2 \|y\|^2 - 4t |\operatorname{Re}(Ax, y)| + 2\alpha \|x\|^2 \geq 0,$$

so that

$$|\operatorname{Re}(Ax, y)| \leq \alpha^2 \|x\|^2 \|y\|^2.$$

Setting $y = Ax$, $\|x\|=1$, we get

$$\|Ax\|^2 \leq \alpha \|Ax\|, \text{ so that}$$

$$\|A\| \leq \alpha.$$

Thus $(B' - B)^3 = 0$ and

$$\|B - B'\|^4 = \|(B - B')^4\| = 0.$$

Def Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$|A| = \sqrt{A^* A}.$$

Def $V \in \mathcal{L}(\mathcal{H})$ is partial isometry (partial unitary), if on $(\text{Ker } V)^\perp$ V is isometry, i.e.,

$$\|Vx\| = \|x\| \quad \forall x \in (\text{Ker } V)^\perp$$

V is partial isometry if

$$V^* V = P \quad \& \quad V V^* = Q$$

for orthogonal projectors P & Q

P - projector onto $(\text{Ker } V)^\perp$,

Q - projector onto $\text{Im } V$. Lecture 23

Thm (Polar decomposition) Let $A \in \mathcal{L}(\mathcal{H})$. Then $\exists!$ partial isometry V s.t. $\text{Ker } V = \text{Ker } A$ and

$$A = V |A|.$$

Moreover, $\text{Im } V = \overline{\text{Im } A}$.

Proof Define $V: \text{Im } |A| \rightarrow \text{Im } A$

by $U(|A|x) = Ax$. Since

$$\| |A|x \|^2 = (|A|^2 x, x) = \| Ax \|^2$$

$\text{Ker } |A| = \text{Ker } A$, so if

$|A|x_1 = |A|x_2$, then $Ax_1 = Ax_2$
and U is well-defined. It
extends to an isometry

$$U: \overline{\text{Im } |A|} \rightarrow \overline{\text{Im } A};$$

set

$$U \left\{ \begin{array}{l} \\ \text{Im } |A|^\perp \end{array} \right. = 0.$$

But $\text{Im } |A|^\perp = \text{Ker } |A| = \text{Ker } A$
($|A|$ is self-adjoint), so that

$$\text{Ker } U = \text{Ker } A$$

Uniqueness is clear: $\text{Ker } U = \text{Ker } A$
 $= \text{Ker } |A| = \text{Im } |A|^\perp$ means that
 U is completely defined by

$$U(|A|x) = Ax.$$

[$U = s$ -limit of polynomials
in A, A^*].

III.2. Compact operators

Def $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is
compact (or completely continuous)
if for every bounded sequence

$\{x_n\} \subset \mathcal{B}_1$, the sequence $\{Tx_n\} \subset \mathcal{B}_2$
has a convergent subsequence.

Equivalently, T is compact, if
it maps bounded sets into precompact
sets.

Examples

1) Finite rank operators - $\dim \text{Im } T < \infty$, i.e.

$$\text{Im } T = \mathbb{C}y_1 \oplus \dots \oplus \mathbb{C}y_n \subset \mathcal{B}_2$$

2) Classical integral operators

$$K \in C([0,1] \times [0,1]),$$

$$\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{G}[0,1] \text{ and}$$

$$(\hat{K}f)(x) = \int_0^1 K(x,y) f(y) dy$$

$$\|\hat{K}\|_\infty \leq \sup_{0 \leq x, y \leq 1} |K(x,y)| = \|K\|_\infty,$$

so \hat{K} is bounded. Since $K(x,y)$

is uniformly continuous, $\forall \varepsilon > 0$:

$$|\hat{K}(f)(x) - \hat{K}(f)(x')|$$

$$< \sup_{y \in [0,1]} |K(x,y) - K(x',y)| \|f\|_\infty$$

$$y \in [0,1]$$

$$\leq \varepsilon \|f\|_\infty \text{ if } |x-x'| < \delta.$$

Thus if $\{f_n\}$ is bounded, the sequence $\{\hat{K}f_n\}$ is uniformly bounded and equicontinuous \Rightarrow by the Ascoli theorem, the sequence $\{\hat{K}f_n\}$ has convergent subsequence.

Lecture 24 10/29/03

Theorem 1. Compact operators map weakly convergent sequences into norm convergent sequences.

Proof Let $x_n \xrightarrow{w} x$; by UBT,

$\|x_n\| \leq C \Rightarrow y_n = Tx_n$ has a convergent subsequence, $\{y_{n_k}\}$. Also, for

$y = Tx$, $y_n \xrightarrow{w} y$, since $\forall l \in \mathcal{B}_2^*$

$$l(y_n) - l(y) = (T'l)(x_n - x) \rightarrow 0.$$

Suppose $y_n \not\rightarrow y$ in norm \Rightarrow

$\exists y_{i_k} \text{ s.t. } \|y_{i_k} - y\| \geq \varepsilon$.

$$k \in \mathbb{N}$$

But y_{i_k} also has a convergent subsequence, it converges to $\tilde{y} \neq y$ and

weakly to y - contradiction.

Thm 2. Let $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$.

(a) If $\{T_n\}$ are compact and $T_n \xrightarrow{v} T$, then T is compact.

(b) T is compact iff T' is compact.

(c) If $S \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$ is compact and $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$, then $ST \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$ is compact.

Proof (a) - diagonal process \Rightarrow

$\{x_n\}$ bounded $\exists \{x_{n_k}\}$ s.t.

$$\|x_n\| \leq C$$

$T_n x_{n_k}$ converges $\forall n$.

Then

$$\|Tx_{n_k} - Tx_{n_e}\| \leq \|(T - T_m)x_{n_k}\|$$

$$+ \|(T - T_m)x_{n_e}\| + \|T_m(x_{n_k} - x_{n_e})\|$$

Choosing m s.t. $\|T - T_m\| < \frac{\epsilon}{3C}$,

we then choose $k, l > N$

$$\text{s.t. } \|T_m(x_{n_k} - x_{n_e})\| < \frac{\epsilon}{3}$$

$$\Rightarrow \|T(x_{n_k}) - T(x_{n_e})\| < \epsilon.$$

Will prove (B) only for $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{H}$.

Lemma $T \in \mathcal{L}(\mathcal{H})$ & T^*T is compact, then T is compact.

Proof Let $\{x_n\} \subset \mathcal{H}$ be bounded,

$$\|T(x_n - x_m)\|^2 = (T^*T(x_n - x_m), x_n - x_m) \leq \|T^*T(x_n - x_m)\| \cdot 2C$$

If $\{T^*T x_{n_k}\}$ is Cauchy, then

$\{Tx_{n_k}\}$ is also Cauchy sequence.

Now if T is compact, then

$T T^* = (T^*)^* T^*$ is compact, so

T^* is also compact.

Summary $\mathcal{J}(\mathcal{H})$ - the set of all compact operators on \mathcal{H} , is closed two-sided ideal in $\mathcal{L}(\mathcal{H})$ (\mathbb{C}^* -algebra, Banach algebra).

Lemma 1 If $T \in \mathcal{L}(\mathcal{H})$ is compact, then $\overline{\text{Im } T}$ is separable.

Proof It is sufficient to show that for every $n \exists x_{n_1}, \dots, x_{n_m}$ s.t

$\forall x, \|x\| = 1, \exists 1 \leq l \leq n$ s.t.

$$\|T(x - x_{l,i})\| < \frac{1}{n}.$$

Suppose that it does not hold $\Rightarrow \exists n \in \mathbb{N}$
 $\exists \{x_m\}$ s.t.

$$\|T(x_m - x_n)\| \geq \frac{1}{n}$$

for all m, m' , $m > m'$, but it contradicts
the fact that T is compact.

Lemma 2 If $T \in \mathcal{L}(\mathcal{H})$ is compact
then $\mathcal{H}/\text{Ker } T$ is separable.

Proof $(\text{Ker } T)^\perp = \overline{\text{Im } T^*}$ &
 T^* is compact.

Thm Let $T \in \mathcal{L}(\mathcal{H})$, T compact (\mathcal{H} is not necessarily separable).

Then $T = \lim_{n \rightarrow \infty} T_n$,

T_n are operators of finite rank.

Proof Let $\{x_n\}$ be the orthonormal
basis for $\mathcal{H}/\text{Ker } T = (\text{Ker } T)^\perp$.

Set

$$\lambda_n = \sup_{\|x\|=1, x \perp \{\text{Ker } T \oplus \mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_n\}} \|Tx\|.$$